9.4. One-dimensional resource distribution problem. Bellman's functional equations

In many real life situations there is a need for the distribution of *n* identical objects in M sectors, which aims at achieving an optimal effect. When it is considered that there is no significance which object ends up in which sector, the distribution is of a one-dimensional resource. For example the distribution of 1000 identical requests to 20 identical servers, searching for hash tables or the carrying out of distributed calculations in a network with the aim of minimal processing time (problems in the field of informatics) or the distribution of 8 identical machines in three departments, the distribution of identical goods among several warehouses and other economy problems with the aim of maximal profit and minimal loss and so on.

A problem of this sort can be solved by artificially representing it as occurring in stages and turning it into a dynamic system. Based on the idea of Bellman's principle starting backwards the given class of problems is solved by finding the optimal policy for every stage. This is achieved by constructing Bellman's so called functional equations.

Example 8. Finding a distribution.

To equip 4 of its offices a company has bought 8 identical computers. A conducted marketing research has established the approximate profit of the firm as a result form the installation of a random number of computers in each of its offices. This data has been filled into the following chart. Find one distribution of the computers and calculate the respective overall profit.

	Computers / Offices	x	0	1	2	3	4	5	6	7	8
		Profits									
x_1	Office 1	$g_1(x_1)$	0	30	50	64	57	52	50	48	42
x_2	Office 2	$g_2(x_2)$	0	50	77	85	88	75	66	51	63
x_3	Office 3	$g_3(x_3)$	0	50	60	65	72	80	56	50	40
x_4	Office 4	$g_4(x_4)$	0	30	44	45	50	55	45	40	55

Solution: Let x_1, x_2, x_3, x_4 stand for the number of computers for individual offices. It is apparent that the requisite for complete distribution is $x_1 + x_2 + x_3 + x_4 = 8$. Let us choose for office 1 a number of computers x_1= 3, which brings a profit of 64. Then for the rest of the offices there need to be distributed a total of 8 - 3 = 5 computers. Let us choose for office 2 x_2 = 2 computers, which bring a profit of 77. There are 3 free computers remaining.

Let’s distribute them x_3 = 1 and x_4 = 2 with respective profits of 50 and 44. The distribution chosen is (x_1, x_2, x_3, x_4) = (3,2,1,2), and the total profit is 64+77+50+44=135.

Note. In this example our goal is not to find the most profitable distribution, but apparently examining all possible variants we could find it. But should the numbers in the problem be greater it becomes apparent that being fully exhaustive is an inefficient method.

One-dimensional resource distribution problem - general formulation

Given are a one-dimensional resource of *m* units and *n* sectors. The prices (profits, losses, time or any other measure) of a given number of units for every sector are known. The problem is to find the distribution of the resource by sectors with the maximal (or minimal) total price.

It is convenient to fill the data into a table of the following type:

	Resource units / Sectors	X	0	1	2	3	4	…	m
		Profits							
x_1	Sector 1	$g_1(x_1)$	$g_1(0)$	$g_1(1)$	$g_1(2)$	$g_1(3)$	$g_1(4)$	…	$g_1(m)$
x_2	Sector 2	$g_2(x_2)$	$g_2(0)$	$g_2(1)$	$g_2(2)$	$g_2(3)$	$g_2(4)$	…	$g_2(m)$
x_3	Sector 3	$g_3(x_3)$	$g_3(0)$	$g_3(1)$	$g_3(2)$	$g_3(3)$	$g_3(4)$	…	$g_3(m)$
...	....	...							
x_n	Sector n	$g_n(x_n)$	$g_n(0)$	$g_n(1)$	$g_n(2)$	$g_n(3)$	$g_n(4)$	…	$g_n(m)$

Symbols used:

x_i - resource units for number *i* sector, *i*=1,2…,*n*;

X - the number of resources to be distributed, i.e. a discrete variable with values 0,1,2,….,*m*;

$g_i(x_i)$ - profits from number *i* sector for x_i resource units.

Then the one-dimensional resource distribution problem looks like this:

Find:

(1) $$L = g_1(x_1) + g_2(x_2) + ... + g_n(x_n) \to \max \ (\min)$$

with the restrictions

(2) $$x_1 + x_2 + ... + x_n = X \ , \quad x_i \geq 0 \, , \ i = 1,2,...,n \, .$$

To solve this problem we will artificially reduce it to taking decisions in stages, looking for the optimal distribution for every stage.

To be more precise:

Stage 1 - optimal distribution for the 1st sector; Stage 2 - optimal distribution for the 2nd sector in relation to the second and so on; nth stage - finding the optimal solution with the optimal profit.

Then Bellman's principle can be applied. To this end the following symbols are introduced:

$f_n(X)$ - profit from the nth stage,

$f_{n-1}(X)$ - profit from (n-1)th stage,

...

$f_1(X)$ - profit from the 1st stage.

Bellman's principle is applied backwards, starting from the optimal solution of the last stage towards the previous optimal solutions. We consider the instance of finding a maximum:

$$
\begin{aligned}
&f_n(X) = \max_{0 \le x_n \le X} \{g_n(x_n) + f_{n-1}(X - x_n)\}, \quad X = 0,1,2,\ldots,m ; \\
&f_{n-1}(X) = \max_{0 \le x_{n-1} \le X} \{g_{n-1}(x_{n-1}) + f_{n-2}(X - x_{n-1})\} \\
&\ldots \\
&f_2(X) = \max_{0 \le x_2 \le X} \{g_2(x_2) + f_1(X - x_2)\} \\
&f_1(X) = \max_{0 \le x_1 \le X} \{g_1(x_1)\}
\end{aligned} \tag{3}
$$

Definition. The point x_i at which is reached $\max f_i(X)$ is called point of intermediate maximum, which will be marked by x_i^* .

The resulting formulas (3) are called **Bellman's functional equations**. They describe the policy chosen at every stage so as to find the solution to problem (1)-(2).

Example 9. A company has bought 5 identical machines which need to be distributed in 3 warehouses. Find the distribution of the machines in the warehouses which gives the maximal total profit using the following table for the profits of every warehouse according to the number of machines:

	Machines / Warehouse	X	0	1	2	3	4	5
		Profits						
x_1	I	$g_1(x_1)$	0	5	8	5	4	7
x_2	II	$g_2(x_2)$	0	10	11	9	10	9
x_3	III	$g_3(x_3)$	0	4	6	8	10	12

Solution: Here n=3, m=5. Using formulas (3) we start backwards to find $f_1(X) = \max\limits_{0 \le x_1 \le X} \{g_1(x_1)\}$. For all values of $X = 0,1,2,...,5$ using the date for warehouse I we calculate respectively

$X = 0$: $f_1(0) = \max\limits_{0 \le x_1 \le 0} \{g_1(x_1)\} = 0$ at $x_1^* = 0$;

$X = 1$: $f_1(1) = \max\limits_{0 \le x_1 \le 1} \{g_1(0), g_1(1-0)\} = \max\limits_{0 \le x_1 \le 1} \{g_1(0), g_1(1)\} = \max\{0, \underline{5}\} = 5$ at $x_1^* = 1$;

$X = 2$:

$$f_1(2) = \max\limits_{0 \le x_1 \le 2} \{g_1(0), g_1(2-1), g_1(2-0)\} = \max\limits_{0 \le x_1 \le 2} \{g_1(0), g_1(1), g_1(2)\} = \max\{0, 5, \underline{8}\} = 8$$

at $x_1^* = 2$;

$X = 3$: $f_1(3) = \max\limits_{0 \le x_1 \le 3} \{g_1(0), g_1(1), g_1(2), g_1(3)\} = \max\{0, 5, \underline{8}, 5\} = 8$ at $x_1^* = 2$;

$X = 4$: $f_1(4) = \max\limits_{0 \le x_1 \le 4} \{g_1(0), g_1(1), g_1(2), g_1(3), g_1(4)\} = \max\{0, 5, \underline{8}, 5, 4\} = 8$,

at $x_1^* = 2$;

$X = 5$: $f_1(5) = \max\limits_{0 \le x_1 \le 5} \{g_1(0), g_1(1), g_1(2), g_1(3), g_1(4), g_1(5)\} = \max\{0, 5, \underline{8}, 5, 4, 5\} = 8$,

at $x_1^* = 2$;

The underlined number is the one which results in max.
The values found this way are filled into a table:

X	0	1	2	3	4	5
$f_1(X)$	0	5	8	8	8	8
x_1^*	0	1	2	2	2	2

The graphs of the functions $f_1(X)$ and $g_1(x_1)$ are shown in fig.7.

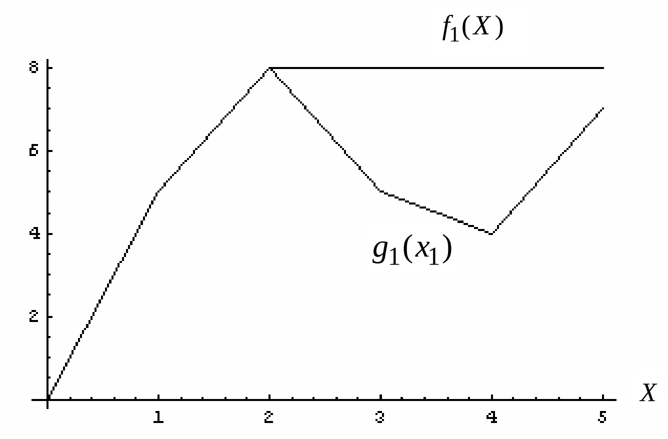

Fig. 7. Graphics of functions $f_1(X)$ and $g_1(x_1)$.

Further on we continue calculating the optimal solution for stage 2, i.e. for

$$f_2(X) = \max_{0 \le x_2 \le X} \{g_2(x_2) + f_1(X - x_2)\}$$

for all values of $X = 0,1,2,...,5$ using the date for warehouse II. We find

$X = 0$: $f_2(0) = \max_{0 \le x_2 \le 0} \{g_2(0) + f_1(0-0)\} = \max\{0,0\} = 0$ at $x_2^* = 0$;

$X = 1$: $f_2(1) = \max_{0 \le x_2 \le 1} \{g_2(0) + f_1(1), g_2(1) + f_1(0)\} = \max\{0+5, \underline{10+0}\} = 10$,

at $x_2^* = 1$;

$X = 2$:

$$f_2(2) = \max_{0 \le x_2 \le 2} \{g_2(0) + f_1(2), g_2(1) + f_1(1), g_2(2) + f_1(0)\} =$$

$$\max\{0+8, \underline{10+5}, 11+0\} = 15$$

at $x_2^* = 1$;

$X = 3$:

$$f_2(3) = \max_{0 \le x_2 \le 3} \{g_2(0) + f_1(3), g_2(1) + f_1(2), g_2(2) + f_1(1), g_2(3) + f_1(0)\} =$$

$$\max\{0+8, \underline{10+8}, 11+5, 9+0\} = 18, \text{ at } x_2^* = 1;$$

$X = 4$:

$$f_2(4) = \max_{0 \le x_2 \le 4} \{g_2(0) + f_1(4), g_2(1) + f_1(3), g_2(2) + f_1(2), g_2(3) + f_1(1), g_2(4) +$$

$$+ f_1(0)\} = \max\{0+8, 10+8, \underline{11+8}, 2+5, 10+0\} = 19$$

at $x_2^* = 2$;

$X = 5$:

$$f_2(5) = \max_{0 \le x_2 \le 5} \{g_2(0) + f_1(5), g_2(1) + f_1(4), g_2(2) + f_1(3), g_2(3) + f_1(2), g_2(4) +$$

$$f_1(1), g_2(5) + f_1(0)\} = \max\{0+8, 10+8, \underline{11+8}, 9+8, 10+5, 9+0\} = 19$$

at $x_2^* = 2$;

Likewise we calculate the remaining function $f_3(X)$ using $f_2(X)$ and the data for $g_3(x_3)$. Then

$X = 0$: $f_3(0) = \max_{0 \le x_3 \le 0} \{g_3(0) + f_2(0)\} = \max\{0,0\} = 0$ at $x_3^* = 0$;

$X = 1$: $f_3(1) = \max_{0 \le x_3 \le 1} \{g_3(0) + f_2(1), g_3(1) + f_2(0)\} = \max\{0+10, 4+0\} = 10$

at $x_3^* = 0$;

$x = 2$:

$$f_3(2) = \max_{0 \le x_3 \le 2} \{g_3(0) + f_2(2), g_3(1) + f_2(1), g_3(2) + f_2(0)\} = \max\{\underline{0+15}, 4+10, 6+0\} = 15$$

at $x_3^* = 0$;

$x = 3$: $f_3(3) = \max\{0+18, \underline{4+15}, 6+10, 8+0\} = 19$ at $x_3^* = 1$;

$x = 4$: $f_3(4) = \max\{0+19, \underline{4+18}, 6+15, 8+10, 10+0\} = 22$ at $x_3^* = 1$;

$x = 5$: $f_3(5) = \max\{0+19, 4+19, \underline{6+18}, 8+15, 10+10, 12+0\} = 24$

at $x_3^* = 2$;

We get the following general table of distributions and profits:

X	0	1	2	3	4	5
$f_1(X)$	0	5	$\boxed{8}$	8	8	8
x_1^*	0	1	2	2	2	2
$f_2(X)$	0	10	15	$\boxed{18}$	19	19
x_2^*	0	1	1	1	2	2
$f_3(X)$	0	10	15	19	22	$\boxed{24}$
x_3^*	0	0	0	1	1	2

<u>Determining optimal distribution</u>

From the table above it becomes obvious that the maximal profit is 24, which is achieved if warehouse III is given 2 machines. The remaining two machines are for the other two warehouses. The maximal profit when $X = 3$ for $f_2(X)$ is 18 for 1 machine in warehouse II. The remaining two machines are in warehouse I with a price of 8. Therefore the maximal profit is achieved by distribution (2,1,2), i.e.:

Warehouse	I	II	III
Number of machines	2	1	2

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